

Do Problems 1-7(b). Problems and parts to problems may be worth different amounts.

Make sure to cite at which points you use the hypothesis of each problem, where you use a previous problem, result, or theorem, etc. As usual, make sure to set up and appropriately end each problem. For problem 7(b), do not need to prove the following:

The Lebesgue Convergence Theorem: $\int_E f_n \rightarrow \int_E f$ if g is integrable over a Lebesgue

measurable set E and $\langle f_n | n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on E such that (i) $f_n \leq g$ on E and (ii) $f_n(x) \rightarrow f(x)$ for almost all $x \in E$.

1. Consider the equivalence classes corresponding to the equivalence relation $\equiv_{\mathbb{Q}}$, where

$x \equiv_{\mathbb{Q}} y \Leftrightarrow x - y \in \mathbb{Q}$. Prove $mE = 0$ if E is a Lebesgue measurable subset of $[0,1]$ that intersects each equivalence class in at most one point.

To receive full credit, indicate which fact(s) in your proof use(s) that E intersects each equivalence class in at most one point. Otherwise, to save time, you may cite "by instructor" for (i) the $E + q$'s are pairwise disjoint for $q \in \mathbb{Q}$ and for (ii) the relation you write down and use for the appropriate closed interval and the union the appropriate $E + q$'s. Of course, you need to state (ii) correctly in order to use it.

2. Prove that if $\langle E_n | n \in \mathbb{N} \rangle$ is a sequence of pairwise disjoint Lebesgue measurable sets, then

$\bigcup_{i \in \mathbb{N}} E_i$ is also Lebesgue measurable.

Feel free to cite and use without proof our Generalized Finite Additivity Property.

3. Let $\langle E_n | n \in \mathbb{N} \rangle$ be a sequence of Lebesgue measurable sets. Prove $m\left(\bigcap_{i \in \mathbb{N}} E_i\right) = \lim_{n \rightarrow \infty} m(E_n)$ if

(i) $\forall n \ E_n \supseteq E_{n+1}$ and (ii) $\exists N \ mE_N < \infty$. Make sure to cite at which points you use (i) and (ii).

4. Suppose $\langle f_n | n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E with finite measure. Also suppose $f_n \rightarrow f$ pointwise. Prove that $\forall \varepsilon > 0 \ \forall \delta > 0 \ \exists \text{ LMS } A \subseteq E \ \exists N \in \mathbb{N}$ such that

(i) $mA < \delta$ and (ii) $\forall x \in E \setminus A \ \forall n \geq N \ |f_n(x) - f(x)| < \varepsilon$.

5. Suppose f is defined and bounded on a Lebesgue measurable set E with finite measure. Prove that if f is a Lebesgue measurable function, then $\inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi = \sup_{\substack{\varphi \leq f \\ \varphi \text{ simple}}} \int_E \varphi$. **Skip the other direction.**

6. Prove the **Bounded Convergence Theorem**: Let $\langle f_n | n \in \mathbb{N} \rangle$ be a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E of finite measure. If $f_n(x) \rightarrow f(x)$ for all $x \in E$ and if the same real number bounds all of the f_n 's, then $\int_E f_n \rightarrow \int_E f$.

7. (a) Prove **Fatou's Lemma**: If $\langle f_n | n \in \mathbb{N} \rangle$ is a sequence of nonnegative Lebesgue measurable functions defined on a Lebesgue measurable set E and if $f_n(x) \rightarrow f(x)$ for almost all $x \in E$, then $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$. Make sure to list the Claim that $h_n \rightarrow h$ pointwise **on the appropriate set**, but feel free to indicate that the Claim holds "by instructor."

(b) The Lebesgue Convergence Theorem follows by applying Fatou to _____ and _____. If not sure about the blanks, you can instead just prove the Lebesgue Convergence Theorem.