

Make sure to set up and appropriately end all proofs.

Textbook: Real Analysis by H. L. Royden, Third Edition

1. Let $x \equiv_{\mathbb{Q}} y$ if and only if $x - y \in \mathbb{Q}$. Prove $\equiv_{\mathbb{Q}}$ is an equivalence relation on $\mathbb{R}$.
2. If $\sim$ is an equivalence relation on X , is then the set of equivalence classes partition X .
3. Consider the equivalence relation $\equiv_{\mathbb{Q}}$. Suppose $f : \{C \mid C \text{ is an equivalence class with respect to } \equiv_{\mathbb{Q}}\} \rightarrow [0, 1)$ is a choice function, i.e., $f(C) \in C$.
 - (a) If $A \subseteq \text{Im}(f)$ and $m(\text{Im}(f)) \downarrow$ where m is the Lebesgue measure (this is $\equiv$ to $\text{Im}(f)$ is Lebesgue measurable), then $m(A) = 0$.
 - (b) Suppose $\forall n \ A \cap \{y \mid n - y \in \mathbb{Q}\}$ has size less than or equal to 1 and A is Lebesgue measurable. Show $m(A) = 0$.
4. Prove m^* is translation invariant, monotonic, and countably subadditive.
5. Prove Corollary 3, Corollary 4, Proposition 5 and problem 8 on p 58.

Corollary 3: If A is countable, $m^*A = 0$.

Corollary 4: The set $[a, b]$ is not countable.

Proposition 5: Given any set A and any $\varepsilon > 0$, there is an open set O such that $A \subset O$ and $m^*O \leq m^*A + \varepsilon$. There is a $G_\varepsilon G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

Problem 8: Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.
6. (a) State the two natural definitions of open (for $\mathbb{R}$). Show they are equivalent.

(b) Do the same for closed.
7. Problem 13 on p 64: Prove proposition 15.

Hints: a. Show that for $m^*E < \infty$, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (vi) (cf. Proposition 5).

b. Use (a) to show that for arbitrary sets E , (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

c. Use (b) to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).

Proposition 15: Let E be a given set. Then the following five statements are equivalent:

- i. E is measurable.
- ii. Given $\varepsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \varepsilon$.
- iii. Given $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.
- iv. There is a G in G_δ with $E \subset G$, $m^*(G \setminus E) = 0$.
- v. There is an F in F_σ with $F \subset E$, $m^*(E \setminus F) = 0$.

If m^*E is finite, the above statements are equivalent to:

- vi. Given $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

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8. Problem 23 on p 71: Prove proposition 22 by establishing the following lemmas:

- Given a measurable function f on $[a, b]$ that takes the values $\pm\infty$ only on a set of measure zero, and given $\varepsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than $\varepsilon/3$.
- Let f be a measurable function on $[a, b]$. Given $\varepsilon > 0$ and M , there is a simple function φ such that $|f(x) - \varphi(x)| < \varepsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take φ so that $m \leq \varphi \leq M$.
- Given a simple function φ on $[a, b]$, there is a step function g on $[a, b]$ such that $g(x) = \varphi(x)$ except on a set of measure less than $\varepsilon/3$. [Hint: Use proposition 15.] If $m \leq \varphi \leq M$, then we can take g so that $m \leq g \leq M$.
- Given a step function g on $[a, b]$, there is a continuous function h such that $g(x) = h(x)$ except on a set of measure less than $\varepsilon/3$. If $m \leq g \leq M$, then we may take h so that $m \leq h \leq M$.

Proposition 22: Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then given $\varepsilon > 0$, we can find a step function g and a continuous function h such that $|f - g| < \varepsilon$ and $|f - h| < \varepsilon$ except on a set of measure less than ε ; i.e.,

$m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

9. Problem 29 on p 73: Give an example to show that we must require $mE < \infty$ in proposition 23.

Proposition 23: Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions defined on E . Let f be real valued function such that for each x in E we have $f_n(x) \rightarrow f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

10. Problem 30 on p 73: Prove Egoroff's Theorem: If $\langle f_n \rangle$ a sequence of measurable functions that converges to a real-valued function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \setminus A$. [Hint: Apply proposition 24 repeatedly with $\varepsilon_n = 1/n$ and $\delta_n = 2^{-n}\eta$.]

Proposition 24: Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions that converges to a real-valued function f a.e. on E . Then, given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and, an N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.