

Quiz 1, MAT 707, Fall 2008, Name_____

Let m be a countably additive, translation invariant measure. Prove that if m is defined on sufficiently many sets (e.g. the σ -algebra on the image E of a choice function from the set of equivalence classes with respect to $\equiv_{\mathbb{Q}}$ into $[0,1)$), then $m[0,1)$ equals either 0 or ∞ . In case you decide to use our three lemmas L_1 , L_2 , and L_3 from class, feel free to cite “by instructor” for their proof. However, you will need to appropriately states the three lemmas in order to use them.

Quiz 2, MAT 707, Fall 2008, Name_____

Prove that $m(\text{Im } f) > 0$ if $f : \{\text{equivalence classes with respect to } \equiv_{\mathbb{Q}}\} \rightarrow [0,1)$ is a choice function, and if $m : \wp(\mathbb{R}) \rightarrow [0, \infty]$ is countably additive, translation invariant, monotone, and defined on all subsets of $\mathbb{R}$, and extends length.

Quiz 3, MAT 707, Fall 2008, Name_____

Prove $m^*([c, d]) \geq d - c$.

Quiz 4, MAT 707, Fall 2008, Name_____

Prove that the outer measure m^* extends length on infinite intervals.

Quiz 5, MAT 707, Fall 2008, Name_____

Prove that for all $a \in \mathbb{R}$, (a, ∞) is a Lebesgue measurable set.

Quiz 6, MAT 707, Fall 2008, Name_____

Prove that if $\langle E_i \mid i \in \mathbb{N} \rangle$ is a series of Lebesgue measurable sets, then $\bigcup_{i \in \mathbb{N}} E_i$ is a Lebesgue

measurable set. You may cite instructor to explain that there exists a Lebesgue measurable set F_i such that the F_i 's are pairwise disjoint and $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} F_i$.

Quiz 7, MAT 707, Fall 2008, Name_____

Assume $\forall i \in \mathbb{N}$ (each E_i is a Lebesgue measurable function) (1) $\exists N$ $mE_N < \infty$, and

(2) $\langle E_n \mid n \in \mathbb{N} \rangle$ is a decreasing sequence, i.e. $E_n \supseteq E_{n+1}$ for all n .

Show $m\left(\bigcap_{i \in \mathbb{N}} E_i\right) = \lim_{k \rightarrow \infty} mE_k$.

Quiz 8, MAT 707, Fall 2008, Name _____

Fill in the blanks.

Suppose f is a Lebesgue measurable function and $g = f$ a.e. To show g is a Lebesgue measurable function, fill in the three blanks appropriately.

$E =$ _____

$$g^{-1}(\alpha, \infty] = \left[f^{-1}(\alpha, \infty] \cup \left(\text{_____} \cap E \right) \right] \setminus \left(\text{_____} \cap E \right).$$

Quiz 9, MAT 707, Fall 2008, Name _____

Proposition 23 (p72)

Suppose $\langle f_n | n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E with finite measure. Also suppose $f_n \rightarrow f$ pointwise (for some real valued function f). Then $\forall \varepsilon > 0 \forall \delta > 0$ there exists a Lebesgue measurable set $A \subseteq E$ such that

(i) $mA < \delta$,

(ii) $\exists N \forall n \geq N \forall x \in E \setminus A, |f_n(x) - f(x)| < \varepsilon$.

Quiz 10, MAT 707, Fall 2008, Name _____

Suppose f is a bounded, Lebesgue measurable function defined on a Lebesgue measurable set E with $mE < \infty$.

Prove $\inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi = \sup_{\substack{\varphi \leq f \\ \varphi \text{ simple}}} \int_E \varphi$.

Quiz 11, MAT 707, Fall 2008, Name _____

Suppose f is a function defined and bounded on a Lebesgue measurable set E with $mE < \infty$.

Prove that if $\inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi = \sup_{\substack{\varphi \leq f \\ \varphi \text{ simple}}} \int_E \varphi$, then f is a Lebesgue measurable function.

Quiz 12, MAT 707, Fall 2008, Name_____

Let f_n is a Lebesgue measurable function such that $f_n \rightarrow f$ pointwise everywhere on $E = \text{dom}(f) = \text{dom}(f_n)$.

Let $h \leq f$ be a bounded Lebesgue measurable function, vanishing outside a set $E' \subseteq E$ such that $m(E') < \infty$. Define $h_n(x)$, so that h_n is a bounded Lebesgue measurable function such that $h_n \leq f_n$.

$$h_n(x) = \underline{\hspace{2cm}}$$

Assume $h_n \rightarrow h$ on E by instructor. Assume h_n is a bounded Lebesgue measurable function by instructor.

Show $\int_E h \leq \lim \int_E f_n$ and give a reason for each step.

- Reasons should include: (1) Bounded Convergence Theorem for one step,
 (2) $h_n \leq f_n$ for another step,
 (3) $f_n \geq 0$ for another step.

$$\int_E h \overset{\substack{\uparrow \\ \text{since } h \\ \text{equals } 0 \\ \text{outside } E'}}{=} \int_{E'} h \overset{\substack{\uparrow \\ \text{(a) why?}}}{=} \underline{\hspace{2cm}} \overset{\substack{\leq \\ \uparrow \\ \text{(b) why?}}}{\leq} \underline{\hspace{2cm}} \overset{\substack{\leq \\ \uparrow \\ \text{(c) why?}}}{\leq} \lim \int_E f_n$$

(a) _____, (b) _____, (c) _____.

Quiz 13, MAT 707, Fall 2008, Name_____

Use Fatou to prove the Lebesgue Convergence Theorem: $\int_E f_n \rightarrow \int_E f$ if g is integrable over a Lebesgue measureable set E and $\langle f_n \mid n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on E such that (i) $f_n \leq g$ on E and (ii) $f_n(x) \rightarrow f(x)$ for almost all $x \in E$.
 As usual, make sure to appropriately set up and end, etc.