

Make sure to cite at which points you use the hypothesis of each problem and where you use a previous theorem. Also make sure to set up and appropriately end problems 1, 2, and 3.

1. (a) Let $\langle E_n \mid n \in \mathbb{N} \rangle$ be a sequence of Lebesgue measurable sets such that:

(i) $\forall n \ E_n \supseteq E_{n+1}$ and (ii) $\exists N \ mE_N < \infty$. Prove that $m\left(\bigcap_{i \in \mathbb{N}} E_i\right) = \lim_{n \rightarrow \infty} m(E_n)$.

(b) Show that (a) may be false without (i).

(c) Show that (a) may be false without (ii).

2. Suppose $\langle f_n \mid n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E with finite measure. Also suppose $f_n \rightarrow f$ pointwise. Prove that $\forall \varepsilon > 0 \ \forall \delta > 0 \ \exists \text{ LMS } A \subseteq E \ \exists N \in \mathbb{N}$ such that

(i) $mA < \delta$ and (ii) $\forall x \in E \setminus A \ \forall n \geq N \ |f_n(x) - f(x)| < \varepsilon$.

3. Suppose f is defined and bounded on a Lebesgue measurable set E with finite measure.

Prove that f is a Lebesgue measurable function $\Leftrightarrow \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi = \sup_{\substack{\varphi \leq f \\ \varphi \text{ simple}}} \int_E \varphi$.

4. (a) Suppose f is such that $\forall \alpha \in \mathbb{R} \ f^{-1}(\alpha, \infty] \in \text{LMS}$ and $\beta \in \mathbb{R}$. Write $f^{-1}[\beta, \infty]$ as either a countable union or countable intersection of sets of the form $f^{-1}(\alpha, \infty]$.

(b) Suppose f and g are Lebesgue measurable functions. Write $(f + g)^{-1}(\alpha, \infty]$ in terms of f^{-1} and g^{-1} of rays using countable unions and intersections so that it is clear that $(f + g)^{-1}(\alpha, \infty]$ is a Lebesgue measurable set.

(c) Suppose f is a Lebesgue measurable function and $g = f$ a.e. and $\beta \in \mathbb{R}$. In class we found Lebesgue measurable sets A , B , and C such that $g^{-1}(\alpha, \infty] = (A \cup B) \setminus C$. List the Lebesgue measurable sets A , B , and C .

(d) Suppose $\langle f_n \mid n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions and $\alpha \in \mathbb{R}$.

Write $\left(\sup_{n \in \mathbb{N}} f_n\right)^{-1}(\alpha, \infty]$ in terms of f_n^{-1} of rays using countable unions and/or intersections so that it is clear that the function $\sup_{n \in \mathbb{N}} f_n$ is a Lebesgue measurable function.