

Problems may be worth different amounts.

As usual, make sure to appropriately set up and end your proofs, make sure to cite any problems results or problems used at the appropriate places, etc.

1. Prove the **Bounded Convergence Theorem**: Let $\langle f_n \mid n \in \mathbb{N} \rangle$ be a sequence of Lebesgue measurable functions defined on a LMS E of finite measure. If $f_n(x) \rightarrow f(x)$ for all $x \in E$ and if the same real number bounds all of the f_n , then $\int_E f_n \rightarrow \int_E f$.

2. Prove **Fatou's Lemma**: If $\langle f_n \mid n \in \mathbb{N} \rangle$ is a sequence of nonnegative Lebesgue measurable functions defined on a LMS E and if $f_n(x) \rightarrow f(x)$ for almost all $x \in E$, then $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$.
 Make sure to show $h_n \rightarrow h$ pointwise on the appropriate set. (Of course, $h_n \rightarrow h$ is probably false.)

3. (i) Prove that if f and g are nonnegative Lebesgue measurable functions defined on a LMS E , then $\int_E (f + g) \leq \int_E f + \int_E g$. Recall here that $\int_E f$ is defined to be the sup of all $\int_E h$ over all bounded LMF $h \leq f$ vanishing outside a set of finite measure.
 (ii) Prove the **Monotone Convergence Theorem**: If $\langle f_n \mid n \in \mathbb{N} \rangle$ is an increasing sequence of nonnegative Lebesgue measurable functions defined on a LMS E and if $f_n(x) \rightarrow f(x)$ for almost all $x \in E$, then $\lim_{n \rightarrow \infty} \int_E f_n$ exists and $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$.
 In Problem #3(ii), increasing does not mean strictly increasing; instead, it means $f_n \leq f_{n+1}$.

4. Prove the following: If f is a nonnegative Lebesgue measurable function defined on and integrable over a LMS E , then $\forall \varepsilon > 0 \exists \delta > 0 \forall A \subseteq E \left(m(A) < \delta \Rightarrow \int_A f < \varepsilon \right)$.

5. Prove the **Lebesgue Convergence Theorem**: $\int_E f_n \rightarrow \int_E f$ if g is integrable over a Lebesgue measurable set E and $\langle f_n \mid n \in \mathbb{N} \rangle$ is a sequence of Lebesgue measurable functions defined on E such that (i) $f_n \leq g$ on E and (ii) $f_n(x) \rightarrow f(x)$ for almost all $x \in E$.
 As usual, make sure to appropriately set up and end, etc.