

1. Provide “elementary” maps that verify the equalities below. Indicate which maps are 1-1 and/or onto, and indicate what bijection verifies the actual equality listed. (Do not forget to take care of 0, and when appropriate, 1, as they are elements of ω .)

1. $|\omega| = |\mathbb{Q}|$

2. $|\omega| = |\mathbb{Q}^5|$

3. $|\omega| = |\mathbb{Q}^{<\omega}|$

2. Provide “elementary” maps that verify the equalities below. Indicate which maps are 1-1 and/or onto, and indicate what bijection verifies the actual equality listed.

1. $|(0,1)| = |\mathbb{R}|$

2. $|(0,1) \cup \{2,3\}| = |\mathbb{R}|$

3. $|(a,b) \dot{\cup} \{x_1, x_2, x_3, \dots\}| = |\mathbb{R}|$

3. (a) Outline the proof of the Bernstein Schroeder Theorem: If $|A| \leq |B|$ and $|B| \leq |A|$,

Let $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} A$

(b) If $b \in B \setminus \text{Im } f$, then $b \in$ _____.

(c) If $a \in A \setminus \text{Im } g$, then $a \in$ _____.

4. (a) Prove $|A| \neq |\wp(A)|$ for any set A .

To receive full credit, show all steps and appropriately set up and end your proof.

(b) Use part (a) to prove $|A| < |\wp(A)|$ for any set A .

To receive full credit, show all steps and appropriately set up and end your proof.

(c) Prove that for any set U there exists a set $E \notin U$.

5. Let $\alpha, \beta \in ON$. Prove that if α is a proper subset of β , then $\alpha \in \beta$.

6. (a) Prove that “ ON satisfies the transitive relation.”

(b) Prove that “ ON satisfies trichotomy.”

(c) Prove that “ ON satisfies the well-ordering principle.”

7. Prove $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for any ordinal α .

Feel free to indicate that the theorem holding for $\alpha = 0$ follows from the techniques of Test 1 and so we may assume $\alpha > 0$.

8. Assume $\alpha > 0$ is a limit ordinal and $\beta : \gamma \xrightarrow[\text{nondecreasing}]{\text{unbounded}} \alpha$. Prove that $cf(\alpha) \geq cf(\gamma)$.

9. Assume that κ is an infinite cardinal. Prove that κ is singular if and only if there exist $\lambda < \kappa$ and $\langle S(\xi) \mid \xi < \lambda \rangle$ such that $|S(\xi)| < \kappa$ and $\kappa = \bigcup_{\xi < \lambda} S(\xi)$.