

MAT 701, Fall 2011 Homework, Name _____

Different problems and parts may not have equal value.

1.

- (a) Suppose $a < b$ and suppose $(a, b) \cap E$ is denumerable. Use Cantor Diagonalization to prove that $(a, b) \setminus E$ is nonempty. To receive full credit, show all steps and appropriately set up and end your proof.
- (b) Use part (a) to show that the set of irrational numbers is dense, i.e. that the set of irrational numbers intersects every nonempty open interval.
- (c) Use part (a) to show that the set of transcendental numbers is dense.
Recall that the set Alg of algebraic numbers over $\mathbb{Q}$ is denumerable and that a real number is transcendental if it is not algebraic.

2. Prove that $\mathbb{Q}^{\mathbb{N}}$ is uncountable.

To receive full credit, show all steps and appropriately set up and end your proof.

3. Recall from the lectures the universal set $O \in \Sigma_1^0 \mid (\omega^\omega)^2$. Let $D = \{g \in \omega^\omega \mid (g, g) \notin O\}$. Prove $D \in \Pi_1^0 \mid \omega^\omega$ by showing that every $g \notin D$ is an interior point of the complement of D , i.e. prove $\forall g \notin D \exists$ an open set $B \subseteq \omega^\omega \setminus D$ such that $g \in B$.

4. Prove that if α is an ordinal, then $S(\alpha) =_{df} \alpha \cup \{\alpha\}$ is also an ordinal.

5. (a) Prove that if X is a set of ordinals (i.e. $\forall \alpha \in X (\alpha \text{ is an ordinal})$), then $\bigcup X =_{df} \{y \mid \exists z \in X (y \in z)\}$ is also an ordinal.

(b) Prove that $\bigcup X$ is $\sup X$. Recalling the definition of sup, part (b) consists of showing that $\bigcup X$ is an upper bound of X and that $\bigcup X$ is the least upper bound:

(b.1) Show $\forall a \in X (a \leq \bigcup X, \text{ i.e. } a \in \bigcup X \text{ or } a = \bigcup X)$.

(b.2) Show that if b is an ordinal such that $\forall a \in X (a \leq b, \text{ i.e. } a \in b \text{ or } a = b)$, then $\bigcup X \leq b, \text{ i.e. } \bigcup X \in b \text{ or } \bigcup X = b$.