

As usual, make sure to appropriately set up and end your proofs, make sure to cite any results or problems used at the appropriate places, etc. Included is a list of results from chapter 5 and 4 to make it easy to cite these appropriately. (Textbook: Real Analysis by H. L. Royden, Third Edition)

1. For any closed interval I with positive length, assume that $L(I)$, $C(I)$ and $R(I)$ form a partition of I into a left closed interval, a center open interval, and a right closed interval, each with positive length, i.e. there exists $a < b < c < d$ such that I is the disjoint union of $L(I) = [a, b]$, $C(I) = (b, c)$, and $R(I) = [c, d]$.

Let $I_\emptyset$ be a fixed closed interval. If $I_{x_1, x_2, \dots, x_n}$ has been defined where $x_1, x_2, \dots, x_n$ is some finite sequence of L 's and R 's, then let $I_{x_1, x_2, \dots, x_n, L} = L(I_{x_1, x_2, \dots, x_n})$ and $I_{x_1, x_2, \dots, x_n, R} = R(I_{x_1, x_2, \dots, x_n})$. Let E_n be the union of the $I_{x_1, x_2, \dots, x_n}$ where each x_i varies over $\{L, R\}$ and let $E = \bigcap_{n=1}^{\infty} E_n$.

Prove that E is perfect (a nonempty closed set in which every point of E is an accumulation point of E). To make the proof easier, assume $\lim_{n \rightarrow \infty} (\text{length}(I_{x_1, x_2, \dots, x_n})) = 0$ for each infinite sequence $(x_1, x_2, x_3, \dots)$ of L 's and R 's. Also feel free to call z a left endpoint if it is the left endpoint of some $I_{x_1, x_2, \dots, x_n}$ and similarly for right endpoint.

2. Do either part (a) or part (b). Only turn in one of these for credit.
 - (a) (Theorem 10) Assume that f is integrable on $[a, b]$ and that $F(x) = F(a) + \int_a^x f(t) dt$. Prove that $F'(x) = f(x)$ for almost all $x \in [a, b]$.
 - (b) (Lemma 13) Prove that if f is absolutely continuous function on $[a, b]$ and $f'(x) = 0$ a.e., then f is constant.
3. (Theorem 14) Prove that every absolutely continuous function F on $[a, b]$ is an indefinite integral of its derivative, i.e. $F(x) = F(a) + \int_a^x F'(t) dt$.
4. For $1 \leq p < \infty$, prove that the L^p spaces are complete.
5. Prove that if $1 \leq p < \infty$ and F is a bounded linear functional on L^p , then $\exists g \in L^q$ such that $F(f) = \int fg$ and $\|F\| = \|g\|_q$.

As with all exam questions, make sure to set up, appropriately finish, cite at which points you use the hypothesis (in particular, make sure to cite at which points you use that F is bounded), etc.

1. **Lemma (Vitali)** (p 98): Let E be a set of finite outer measure and J a collection of intervals that cover E in the sense of Vitali. Then, given $\varepsilon > 0$, there is a finite disjoint collection $\{I_1, \dots, I_N\}$ of intervals in J such that $m^* \left[E \setminus \bigcup_{n=1}^N I_n \right] < \varepsilon$.
2. **Proposition** (p 99): If f is continuous on $[a, b]$ and one of its derivatives (say D^+) is everywhere nonnegative on (a, b) , then f is nondecreasing on $[a, b]$; i.e., $f(x) \leq f(y)$ for $x \leq y$.
3. **Theorem** (p 100): Let f be an increasing real-valued function on the interval $[a, b]$. Then f is differentiable almost everywhere. The derivative f' is measurable, and $\int_a^b f'(x) dx \leq f(b) - f(a)$.
4. **Lemma** (p 103): If f is of bounded variation on $[a, b]$, then $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$.
5. **Theorem** (p 103): A function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$.
6. **Corollary** (p 104): If f is of bounded variation on $[a, b]$, then $f'(x)$ exists for almost all x in $[a, b]$.
7. **Lemma** (p 105): If f is integrable on $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t) dt$ is a continuous function of bounded variation on $[a, b]$.
8. **Lemma** (p 105): If f is integrable on $[a, b]$ and $\int_a^x f(t) dt = 0$ for all x in $[a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.
9. **Lemma** (p 106): If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t) dt + F(a)$, then $F'(x) = f(x)$ for almost all x in $[a, b]$.
10. **Theorem** (p 107): Let f be integrable function on $[a, b]$, and suppose that $F(x) = F(a) + \int_a^x f(t) dt$. Then $F'(x) = f(x)$ for almost all x in $[a, b]$.
11. **Lemma** (p 108): If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.
12. **Corollary** (p 109): If f is absolutely continuous, then f has a derivative almost everywhere.
13. **Lemma** (p 109): If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is constant.
14. **Theorem** (p 110): A function F is an indefinite integral if and only if it is absolutely continuous.
15. **Proposition (from chapter 4.14)** (p 88): Let f be a nonnegative function which is integrable over a set E . Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have $\int_A f < \varepsilon$.