

Directions: Make sure to set up and end your proofs appropriately.

1. For this problem, we consider a generalization of the Cantor set. We construct a set E similarly to the construction of the Cantor set, except:
 - (i) the open intervals we remove need not be middle thirds (possibly $3/100$ is the ratio of lengths between one open interval and the closed interval from which it is being removed, and $\frac{1}{\pi}$ is another such ratio),
 - (ii) each open interval that we remove from a closed interval need not be centered around the midpoint of the closed interval, though we require the open interval to contain the center of the closed interval and “leave” two closed intervals of positive length.

Precise construction of E : Let $F_\emptyset = [0,1]$. Suppose F_L and F_R are left and right closed intervals, each with positive length, obtained by removing an open interval $I_\emptyset \subset F_\emptyset$ from $F_\emptyset$ such that $\frac{1}{2} \in I_\emptyset$.

Similarly let $F_{L,L}$ and $F_{L,R}$ respectively be the left and right closed intervals, each with positive length, obtained by removing from F_L some open interval $I_L \subseteq F_L$ which contains the midpoint of F_L .

Let $F_{R,L}$ and $F_{R,R}$ be the left and right closed intervals, each with positive length, obtained by removing some open interval $I_R \subseteq F_R$ from F_R which contains the midpoint of F_R . In general, if $F_{x_1, x_2, \dots, x_n}$ has been defined where $x_1, x_2, \dots, x_n$ is some finite sequence of L 's and R 's, then let $F_{x_1, x_2, \dots, x_n, L}$ and $F_{x_1, x_2, \dots, x_n, R}$ be the left and right closed intervals, each with positive length, obtained by removing from $F_{x_1, x_2, \dots, x_n}$ some open interval $I_{x_1, x_2, \dots, x_n} \subseteq F_{x_1, x_2, \dots, x_n}$ which contains the midpoint of $F_{x_1, x_2, \dots, x_n}$. Let E_n be the union of the $F_{x_1, x_2, \dots, x_n}$

where each x_i varies over $\{L, R\}$ and let $E = \bigcap_{n=1}^{\infty} E_n$.

(a) Prove that E is closed and compact.

(b) Prove that the closure of E (which equals E by part (a)) does not contain any nonempty open interval (a, b) and prove that the complement of the closure of E is dense. Either of these can be taken as the definition of nowhere dense (see page 158). Be careful: In the case of the Cantor set C , this problem is trivial since $mC = 0$, but E may have positive measure.

(c) Prove that E is uncountable. For each infinite sequence $(x_1, x_2, x_3, \dots)$ of L 's and R 's, feel free to

assume $\bigcap_{n=1}^{\infty} F_{x_1, x_2, \dots, x_n}$ is nonempty and therefore contains a point $y_{(x_1, x_2, x_3, \dots)}$.

(d) Prove that $mC = 0$ and then use this fact (instead of part c above) to prove that the Cantor set contains no nonempty open interval and that the Cantor set is nowhere dense.

(e) Do problem 14(b) on page 64. You may use the parts (a) through (d) above if you choose.

Let F be a subset of $[0,1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the n th step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is a closed set, $\tilde{F}$ dense in $[0,1]$ and $mF = 1 - \alpha$. Such a set F is called a *generalized Cantor set*.

2. Prove that the set E above is perfect.

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3. Suppose $E \subseteq \mathbb{R}$ is arbitrary.

(a) Show that if E is closed, then E' is closed.

(b) Show that if E is closed, then E^α is closed for all α , where $E^0 = E$, $E^{\alpha+1} = (E^\alpha)'$, and

$$E^\alpha = \bigcap_{\beta < \alpha} E^\beta \text{ for } \alpha \text{ a limit ordinal.}$$

4. Directly show that every perfect set is uncountable.