

Directions: Make sure to set up and end your proofs appropriately.

- Recall our proof of the Cantor-Bendixson Theorem (also called the CUP Theorem according to the internet) and recall the function $n: \bigcup_{\alpha < \theta(F)} (F^\alpha \setminus F^{\alpha+1}) \rightarrow \mathbb{N}$ defined by $n(x) = \mu n(I_n \cap F^\alpha = \{x\})$, in which F is a closed set, $I_1, I_2, I_3, \dots$ is an enumeration of the open intervals with rational endpoints, and $\theta(F)$ is the least ordinal θ for which $F^\theta \setminus F^{\theta+1} = \emptyset$. Prove that n is injective.
- Prove that the Cantor Set is uncountable using the “middle-thirds interval” definition of the Cantor Set, not the definition in terms of the ternary expansion.
- Let $\langle f_n | n \in \mathbb{N} \rangle$ be a sequence of Lebesgue measurable functions that converge in measure to f . Prove that a subsequence $\langle f_{n(k)} | k \in \mathbb{N} \rangle$ exists that converges to f almost everywhere.
- Let E be a set of finite outer measure contained in an open set O and let J be a Vitali covering of E which contains only closed sets that are subsets of O . Pick an arbitrary $\varepsilon > 0$. Recall the definition of k_n , the construction of the pairwise disjoint $I_1, I_2, I_3, \dots \in J$ such that $lh(I_{n+1}) > \frac{k_n}{2}$ and that N was chosen so that $\sum_{i>N} lh(I_i) < \frac{\varepsilon}{5}$. Prove $E \setminus \bigcup_{i \leq N} I_i \subseteq \bigcup_{i>N} (5I_i)$.
- Assume that f is an increasing real-valued function on $[a, b]$. Let $E_{u,v} = \{x : D^+ f(x) > u > v > D_- f(x)\}$. Prove $m^* E_{u,v} = 0$. Feel free to skip the construction of $I_1, I_2, I_3, \dots, I_N$ which are pairwise disjoint elements of the Vitali Covering J and assume $m^*\left(E_{u,v} \setminus \bigcup_{i \leq N} I_i\right) < \varepsilon$,

$$m^*\left(\bigcup_{i \leq N} I_i^O\right) > m^*(E_{u,v}) - \varepsilon \quad (**), \text{ and}$$

$$\sum_{i=1}^N (f(x_i) - f(x_i - h)) < v \sum_{i=1}^N h_i < v (m^*(E_{u,v}) + \varepsilon) \quad (*).$$
- Assume that f is of bounded variation on $[a, b]$. Prove that f is the difference of two monotone real-valued functions on $[a, b]$.

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7. Assume that f is integrable on $[a, b]$ and $f > 0$ on the closed set $F \subseteq [a, b]$ with positive measure.

Show that there exists $x \in [a, b]$ such that $\int_a^x f \neq 0$.

8. **Lemma 9** (p106): If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t) dt + F(a)$, then

$F'(x) = f(x)$ for almost all x in $[a, b]$.

9. **Lemma 13** (p109): If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is constant.