

Directions: Make sure to set up and end your proofs appropriately.

For any closed interval I with positive length, assume that $L(I)$, $C(I)$ and $R(I)$ form a partition of I into a left closed interval, a center open interval, and a right closed interval, each with positive length, i.e. there exists $a < b < c < d$ such that I is the disjoint union of $L(I) = [a, b]$, $C(I) = (b, c)$, and $R(I) = [c, d]$.

Let $I_\emptyset$ be a fixed closed interval. If $I_{x_1, x_2, \dots, x_n}$ has been defined where $x_1, x_2, \dots, x_n$ is some finite sequence of L 's and R 's, then let $I_{x_1, x_2, \dots, x_n, L} = L(I_{x_1, x_2, \dots, x_n})$ and $I_{x_1, x_2, \dots, x_n, R} = R(I_{x_1, x_2, \dots, x_n})$. Let E_n be the union of the $I_{x_1, x_2, \dots, x_n}$ where each x_i varies over $\{L, R\}$ and let $E = \bigcap_{n=1}^{\infty} E_n$.

1. Prove that E is uncountable using a Cantor Diagonalization argument. For each infinite sequence $(x_1, x_2, x_3, \dots)$ of L 's and R 's, feel free to assume $\bigcap_{n=1}^{\infty} I_{x_1, x_2, \dots, x_n}$ is nonempty and therefore contains a point $y_{(x_1, x_2, x_3, \dots)}$, but do not assume the $y_{(x_1, x_2, x_3, \dots)}$'s are different for different infinite sequences $(x_1, x_2, x_3, \dots)$. Instead show this as needed.
2. Prove that E is perfect (a nonempty closed set in which every point of E is an accumulation point of E). To make the proof easier, assume $\lim_{n \rightarrow \infty} (\text{length}(I_{x_1, x_2, \dots, x_n})) = 0$ for each infinite sequence $(x_1, x_2, x_3, \dots)$ of L 's and R 's. Also feel free to call z a left endpoint if it is the left endpoint of some $I_{x_1, x_2, \dots, x_n}$ and similarly for right endpoint.
3. Consider the special case in which E is the Cantor Set so that $I_\emptyset = [0, 1]$ and $C(I_{x_1, x_2, \dots, x_n})$ is the middle-thirds open interval of $I_{x_1, x_2, \dots, x_n}$.
 - (a) Suppose $y = \sum_{i \in \mathbb{N}} \frac{x_i}{3^i}$ for some sequence $(x_1, x_2, x_3, \dots)$ of 0's and 2's. Prove that $y \in E_3$. Feel free to use that the length of each I_{x_1, x_2, x_3} is $\frac{1}{27}$ in this special case of the Cantor Set.
 - (b) Let $(x_1, x_2, x_3, \dots) = (L, R, L, R, L, R, \dots)$ and $y \in I_{x_1, x_2, \dots, x_n}$ for each n . Calculate the rational number y and write it as a fraction.
4. Suppose C is an uncountable closed set. Prove that C contains a perfect subset (a nonempty closed set in which every point of E is an accumulation point of E).