

Results covered in class:

Lemma 8.2 page 91. Let κ be a regular uncountable cardinal. Then the intersection of two club subsets of κ is also club.

Theorem 8.3 page 92. If κ is a regular uncountable cardinal, $\lambda < \kappa$, and $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a sequence of club subsets of κ , then $\bigcap_{\alpha < \lambda} C_\alpha$ is club.

Corollary to 8.3. The club filter on a regular uncountable cardinal κ is κ -complete.

Lemma 8.4 page 92. If κ is a regular uncountable cardinal and $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a sequence of club subsets of κ , then $\bigtriangleup_{\alpha < \kappa} C_\alpha$ is also club.

Corollary 8.5 page 93. The club filter on a regular uncountable cardinal κ is closed under diagonal intersections.

Theorem 8.7 Fodor's Theorem page 93. Let κ be a regular uncountable cardinal. If f is a regressive function on a stationary set $S \subseteq \kappa$, then there exist stationary $T \subseteq S$ such that f restricted to T is constant.

Page 94: The club filter on any cardinal $\kappa \geq \omega_2$ is not an ultrafilter (consider E_ω^κ and $E_{\omega_1}^\kappa$).
Using AC and 8.10 below, the club filter on ω_1 is not an ultrafilter.

Lemma 8.8. page 94. For each regular uncountable cardinal κ , every stationary subset of E_ω^κ is the union of κ disjoint stationary sets.

Page 94. For each regular uncountable cardinal κ and each cardinal $\lambda \in (\omega, \kappa)$, every stationary subset of E_λ^κ is the union of κ disjoint stationary sets and (using Fodor's Theorem) every stationary subset of $\{\alpha < \kappa \mid cf(\alpha) < \alpha\}$ is the union of κ disjoint stationary sets.

Lemma 8.9 page 94. If S is a stationary subset of regular uncountable cardinal κ such that every $\alpha \in S$ is a regular uncountable cardinal, then $T = \{\alpha \in S \mid S \cap \alpha \text{ is not a stationary subset of } \alpha\}$ is a stationary subset of κ .

Theorem 8.10 (Solovay) page 95. Every stationary subset of a regular uncountable cardinal κ is the disjoint union of κ stationary subsets.

Page 95.

- (1) If κ is an inaccessible cardinal, then $\{\alpha < \kappa \mid \alpha \text{ is a strong limit cardinal}\}$ is club.
- (2) If κ is the least inaccessible cardinal, then all strong limit cardinals below κ are singular so that $\{\alpha < \kappa \mid \alpha \text{ is a singular strong limit cardinal}\}$ is club.

- (3) If $\alpha < \kappa$ and κ is the α^{th} inaccessible cardinal, then $\{\beta < \kappa \mid \beta \text{ is a regular cardinal}\}$ is nonstationary.
- (4) If κ is a Mahlo cardinal (i.e. inaccessible and $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary), then $\{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is stationary and κ is the κ^{th} inaccessible cardinal.

Theorem 9.1 Ramsey page 108. For all $n < \omega$ and all $k < \omega$, $\aleph_0 \rightarrow (\aleph_0)_k^n$, i.e. for every $F : [\omega]^n \rightarrow k$ there exists an infinite $H \subseteq \omega$ such that F restricted to $[H]^n$ is constant.

Lemma 9.3 page 110. $2^\kappa \not\rightarrow (\omega)_\kappa^2$ for every infinite cardinal κ . In fact, the most natural partition of $[2^\kappa]^2$ into κ pieces does not have a homogeneous set of size 3 (let alone ω).

Lemma 9.4. page 110. $2^\kappa \not\rightarrow (\kappa^+)_2^2$ for every infinite cardinal κ . Therefore, $\aleph_1 \not\rightarrow (\aleph_1)_2^2$ and so the obvious generalization of Ramsey's Theorem is false.

Lemma 9.5. page 110. The lexicographically ordered set $\{0,1\}^\kappa$ has no increasing or decreasing κ^+ -sequence. (This is used to prove Lemma 9.4.)

Discussed Ramsey cardinals from page 121.

Lemma 8.11 page 96. If κ is regular and uncountable and if F is a normal filter that contains (as elements) all final segments (α, κ) , then F contains all club subset of κ .

Lemma 10.4 page 127. Every measurable cardinal is inaccessible (i.e. regular and a strong limit cardinal).

Lemma 10.19. page 134 If D is a normal measure on κ (i.e. a κ -complete nonprincipal ultrafilter closed under diagonal intersections), then every set in D is stationary.

Theorem 10.20 page 134. Every measurable cardinal carries a normal measure. If U is a nonprincipal κ -complete ultrafilter on κ , then there exists a function $f : \kappa \rightarrow \kappa$ such that $f_*(U) = \{X \subseteq \kappa \mid f^{-1}(X) \in U\}$ is a normal measure.

Theorem 10.21 page 135. Every measurable cardinal is a Mahlo cardinal (i.e. inaccessible and $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary). In fact, $\{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is an element of the normal measure and therefore stationary by 10.19.

Theorem 10.22. page 136. Every measurable cardinal is a Ramsey cardinal. In fact, if D is a normal measure on a measurable cardinal κ , then for every $F : [\kappa]^{<\omega} \rightarrow \lambda$ with $\lambda < \kappa$ there exists $H \in D$ that is homogeneous for F (i.e. $\forall n < \omega \exists i_n < \lambda$ F only takes the constant value i_n on $[H]^n$).