

Spring 2010 Math 702 TEST 2

As always, make sure to cite any hypotheses, any previous results or problems used, etc. at the appropriate places in you proofs. Also, make sure to appropriately start and end your proofs.

Set up for problem #1 and #2: Let κ be a regular, uncountable cardinal, let A be a stationary subset of κ , and let $A^L = \{\alpha \in A \mid \alpha \text{ is a limit ordinal (in particular infinite)}\}$. By work done in problem #3 below, one of the two sets $A^S = \{\alpha \in A^L \mid \text{cf}(\alpha) < \alpha\}$ and $A^R = \{\alpha \in A^L \mid \alpha \text{ is a regular uncountable cardinal}\}$ is a stationary subset of κ . Assume A^R is stationary, in which case one can show

$W = \{\alpha \in A^R \mid \alpha \cap A^R \text{ is not a stationary subset of } \alpha\}$ is a stationary subset of κ . Since

$\forall \alpha \in W \exists K^\alpha (K^\alpha \text{ is club in } \alpha \text{ and disjoint from } \alpha \cap A^R)$, for each $\alpha \in W$ let $f^\alpha : \alpha \xrightarrow[\text{continuous}]{\text{unbounded}} K^\alpha$ enumerate K^α .

1. Prove that following Main Lemma: there exists $i < \kappa$ such that for all $\eta < \kappa$, we have

$F_i^\eta =_{df} \{\alpha \in W \mid \alpha > i \text{ and } f^\alpha(i) \geq \eta\}$ is stationary.

2. Using the Main Lemma in problem #1, prove that W is the union of κ pairwise disjoint stationary sets. (Note that this is similar to quiz 5 but different in that the W in quiz 5 is a stationary subset of E_ω^κ .)

3. Let κ be a regular, uncountable cardinal, and let A be a stationary subset of κ . Prove that A is the union of κ pairwise disjoint stationary sets.

Hints: (a) By the HW, assume that $A^L = \{\alpha \in A \mid \alpha \text{ is a limit ordinal (in particular infinite)}\}$ is stationary.

(b) Show that one of two sets $A^S = \{\alpha \in A^L \mid \text{cf}(\alpha) < \alpha\}$ and

$A^R = \{\alpha \in A^L \mid \alpha \text{ is a regular uncountable cardinal}\}$ is stationary.

(c) If A^R is stationary, use problem 2.

(d) If A^S is stationary, use Fodor's Lemma, and feel free to cite that for every infinite $\lambda < \kappa$ every stationary subset of E_λ^κ is the union of κ pairwise disjoint stationary sets.

(e) Use that from (c) and (d), you now have κ pairwise disjoint stationary subsets of A ; from this, get that the original A is the union of κ pairwise disjoint stationary sets.

4. Prove Ramsey's Theorem that $\aleph_0 \rightarrow (\aleph_0)_k^n$ for all $n < \omega$ and all $k < \omega$.