

Math 751 Spring 2011 Test 1 questions.

1. Quiz 1:

Prove the existence case of Gödel's Normal Form Theorem (GNFT). In this case, $\varphi(u_1, u_2, \dots, u_n)$ is the formula $(\exists u_{n+1} \in u_i) \psi(u_1, u_2, \dots, u_{n+1})$.

2. Quiz 2:

Prove the conjunction $\wedge$ and negation $\neg$ cases of Gödel's Normal Form Theorem.

Recall Lemma 13.7: If G is a Gödel operation, then the property $Z = G(X_1, \dots, X_n)$ can be written as a Δ_0 -formula. In fact,

- (i) $u \in G(X, \dots)$ can be written as a Δ_0 -formula φ_G^∞ .
- (ii) If ψ is Δ_0 , then so are $\forall u \in G(X, \dots) \psi$ and $\exists u \in G(X, \dots) \psi$ (maybe respective Δ_0 -formula could be named $\varphi_{\forall, G}$ and $\varphi_{\exists, G}$).
- (iii) $Z = G(X, \dots)$ can be written as a Δ_0 -formula φ_G^∞ .
- (iv) If ψ is Δ_0 , then $\psi(G(X, \dots))$ can be written as a Δ_0 -formula $\varphi_{\psi, G}$.

3. Quiz 3:

Prove (iii) and (iv) from (i) and (ii).

4. Quiz 4:

Prove (i).7 and (ii).7, i.e., prove (i) and (ii) for the $G = G_7$ assuming as IH that (i), (ii), (iii), (iv) hold for a Gödel operation of less complexity than G .

Gödel's Normal Form Theorem (13.4): There exist operations $G_1, G_2, \dots, G_{10}$ such that if $\varphi(u_1, u_2, \dots, u_n)$ is a Δ_0 -formula, then there exists a composition G_φ of $G_1, G_2, \dots, G_{10}$ such that for all $X_1, \dots, X_n$, $G(X_1, \dots, X_n) = \{(u_1, \dots, u_n) \mid u_1 \in X_1, \dots, u_n \in X_n \text{ and } \varphi(u_1, \dots, u_n)\}$.

In the inductive proof, recall wlog,

- (i) the only logical symbols in φ are $\neg$, $\wedge$, and restricted $\exists$
- (ii) $=$ does not occur
- (iii) the only occurrence of $\in$ is $u_i \in u_j$ where $i \neq j$
- (iv) the only occurrence of $\exists$ is $(\exists u_{m+1} \in u_i) \psi(u_1, u_2, \dots, u_{m+1})$ where $i \leq m$.

Gödel operations

$$G_1(X, Y) = \{X, Y\}$$

$$G_2(X, Y) = X \times Y$$

$$G_3(X, Y) = \varepsilon(X, Y) = \{(u, v) \mid u \in X \wedge v \in Y \wedge u \in v\}$$

$$G_4(X, Y) = X - Y$$

$$G_5(X, Y) = X \cap Y$$

$$G_6(X) = \bigcup X$$

$$G_7(X) = \text{dom } X$$

$$G_8(X) = \{(u, v) \mid (v, u) \in X\}$$

$$G_9(X) = \{(u, v, w) \mid (u, w, v) \in X\}$$

$$G_{10}(X) = \{(u, v, w) \mid (v, w, u) \in X\}$$