

Quiz1: Assume α and γ are limit ordinals such that $\exists \beta: \gamma \xrightarrow[\text{nondecreasing}]{\text{unbounded}} \alpha$. Prove that $cf(\alpha) \leq cf(\gamma)$.

Feel free to cite HW#3 that a suitable composition of unbounded, nondecreasing maps is unbounded. However, otherwise complete the proof; in particular, prove that the skipping function actually works (has domain equal to some ordinal $\leq cf(\gamma)$, etc.) Set up and end the proof appropriately.

Quiz 2: Assume α and γ are limit ordinals such that $\exists \beta: \gamma \xrightarrow[\text{nondecreasing}]{\text{unbounded}} \alpha$. Prove that $cf(\gamma) \leq cf(\alpha)$.

Quiz 3: Prove that if κ is an infinite cardinal, then $\kappa < \kappa^{cf(\kappa)}$.

$\kappa^{cf(\kappa)}$ Involves cardinal exponentiation and is therefore a cardinal. It is the cardinal $\left| {}^{cf(\kappa)}\kappa \right|$ where ${}^{cf(\kappa)}\kappa$ is the set of all functions from $cf(\kappa)$ into κ .

Quiz4: Assume that κ is an infinite cardinal such that there exist $\lambda < \kappa$ and $\langle S(\xi) \mid \xi < \lambda \rangle$ such that $|S(\xi)| < \kappa$ and $\kappa = \bigcup_{\xi < \lambda} S(\xi)$. Prove that κ is singular, i.e. show that $cf(\kappa) < \kappa$.

Note: There is no Quiz 5.

Quiz6: Prove $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for any ordinal α . Feel free to assume the base step for $\alpha = 0$ (by instructor).

Quiz7: Let κ be a regular, uncountable cardinal and $\langle C_i \mid i \in I \rangle$ be a sequence of closed subsets of κ . Show $\bigcap_{i \in I} C_i$ is also closed.

Quiz8: Let κ be a regular uncountable cardinal.

- Prove that the intersection of two club subsets of κ is also club.
- Give an example that verifies the intersection of two unbounded subsets of κ need not be unbounded.

Quiz9: Let κ be a regular uncountable cardinal.

- Prove that if $\lambda < \kappa$ and $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a sequence of club subsets of κ , then $\bigcap_{\alpha < \lambda} C_\alpha$ is club.
- Give an example that verifies that part (a) is false if " $\lambda < \kappa$ " is replaced by $\lambda = \kappa$.

Quiz10: Let κ be a regular, uncountable cardinal and $\langle C_\alpha \mid \alpha < \kappa \rangle$ be a sequence of club subsets of κ . Show

$\Delta_{\alpha < \kappa} C_\alpha$ is also club.

Quiz11: Prove Fodor's Theorem: Let κ be a regular uncountable cardinal. If f is a regressive function on a stationary set $S \subseteq \kappa$, then there exists a stationary $T \subseteq S$ such that f restricted to T is constant.

Quiz12: Recall the following theorem: Let κ be a regular, uncountable cardinal. Every stationary subset of E_ω^κ is the union of κ pairwise disjoint stationary sets. State and prove the Main Lemma to this theorem.

Quiz 13: Let κ be a regular, uncountable cardinal and let W be a stationary subset of E_ω^κ . For each $\alpha \in W$, let

$f_\alpha: \omega \xrightarrow[\text{unbounded}]{\text{increasing}} \alpha$. Assume the Main Lemma: there exists $i < \omega$ such that for all $\eta < \kappa$, we have

$W_{\geq \eta}^i =_{df} \{\alpha \in W \mid f^\alpha(i) \geq \eta\}$ is stationary. Using the Main Lemma, prove that W is the union of κ pairwise disjoint stationary sets.

Quiz 14: Let κ be a regular, uncountable cardinal, let A be a stationary subset of κ , and let

$A^L = \{\alpha \in A \mid \alpha \text{ is a (infinite) limit ordinal}\}$. Recall that it is easy to show that one of two sets

$A^S = \{\alpha \in A^L \mid \text{cf}(\alpha) < \alpha\}$ and $A^R = \{\alpha \in A^L \mid \alpha \text{ is a regular uncountable cardinal}\}$ is stationary. Assume A^R is

stationary, in which case one can show $W = \{\alpha \in A^R \mid \alpha \cap A^R \text{ is not a stationary subset of } \alpha\}$ is a stationary subset of

κ so that $\forall \alpha \in W \exists F_\alpha (F_\alpha \text{ is club in } \alpha \text{ and disjoint from } \alpha \cap A^R)$. For each $\alpha \in W$, let $f_\alpha : \alpha \xrightarrow[\text{continuous}]{\text{unbounded}} F_\alpha$

enumerate F_α . Prove that following Main Lemma: there exists $\xi < \kappa$ such that for all $\eta < \kappa$, we have

$W_{\geq \eta}^\xi =_{df} \{\alpha \in W \mid \alpha > \xi \text{ and } f_\alpha(\xi) \geq \eta\}$ is stationary.