

Directions: Make sure to set up and end your proofs appropriately.

1. Prove that if κ is an infinite cardinal, then $\kappa < \kappa^{cf(\kappa)}$. $\kappa^{cf(\kappa)}$ involves cardinal exponentiation and is therefore a cardinal. It is the cardinal $\left| {}^{cf(\kappa)}\kappa \right|$ where ${}^{cf(\kappa)}\kappa$ is the set of all functions from $cf(\kappa)$ into κ .
2. Let κ be a regular, uncountable cardinal and let W be a stationary subset of E_ω^κ . For each $\alpha \in W$, let $f_\alpha : \omega \xrightarrow[\text{unbounded}]{\text{increasing}} \alpha$. Assume the Main Lemma: there exists $i < \omega$ such that for all $\eta < \kappa$, we have $W_{\geq \eta}^i =_{df} \{\alpha \in W \mid f_\alpha(i) \geq \eta\}$ is stationary. Using the Main Lemma, prove that W is the union of κ pairwise disjoint stationary sets.
3. Let κ be a regular, uncountable cardinal, let A be a stationary subset of κ , and let $A^L = \{\alpha \in A \mid \alpha \text{ is a (infinite) limit ordinal}\}$. Recall that it is easy to show that one of two sets $A^S = \{\alpha \in A^L \mid cf(\alpha) < \alpha\}$ and $A^R = \{\alpha \in A^L \mid \alpha \text{ is a regular uncountable cardinal}\}$ is stationary. Assume A^R is stationary, in which case one can show $W = \{\alpha \in A^R \mid \alpha \cap A^R \text{ is not a stationary subset of } \alpha\}$ is a stationary subset of κ so that $\forall \alpha \in W \exists F_\alpha (F_\alpha \text{ is club in } \alpha \text{ and disjoint from } \alpha \cap A^R)$. For each $\alpha \in W$, let $f_\alpha : \alpha \xrightarrow[\text{continuous}]{\text{unbounded}} F_\alpha$ enumerate F_α . Prove that following Main Lemma: there exists $\xi < \kappa$ such that for all $\eta < \kappa$, we have $W_{\geq \eta}^\xi =_{df} \{\alpha \in W \mid \alpha > \xi \text{ and } f_\alpha(\xi) \geq \eta\}$ is stationary.